



**NPTEL ONLINE CERTIFICATION COURSES**

# **FINITE ELEMENT METHOD**

**Biswanath Banerjee & Amit Shaw**

**Department of Civil Engineering, IIT Kharagpur**

**Module 03: Weighted Residual Method and 1D Bar Problem**

**Lecture 12 : Weighted Residual Method**

# Concepts Covered

- **Weighted Residual Methods**
- **Galerkin Formulations**
- **Petrov-Galerkin Method**
- **Least Square Method**
- **Collocation Method**





/ weighted Integral statement for PDE

/ weak Form / variational statement.

/  $B(w, u) = L(w)$ ,  $u = \sum_{i=1}^n c_i \phi_i(x) + \phi_0$ ,  $w = \phi_i$

/  $w$  should satisfy homogeneous BC.  $w = \delta u$

/  $I(u) = \frac{1}{2} [B(u, u) - L(u)]$

/ weighted Residual Method.

$Au = f$  with appropriate BC.

Approx.  $u_{\text{approx}} := \tilde{u}$

Residual  $\rightarrow R = A\tilde{u} - f$

Weighted Residual statement

Error Orthogonalization

$u \cdot v = 0$



$\int_{\Omega} w (A\tilde{u} - f) d\Omega = 0 \Rightarrow \int w R d\Omega = 0$

↓  
Error

$A \equiv \frac{d^2}{dx^2}$

$A \equiv \frac{d^2}{dx^2} + \sin \leftarrow \text{nonlinear}$

$$\int w R dx = 0$$

$$R = A\tilde{u} - f, \quad \tilde{u} = \sum_{j=1}^N c_j \underbrace{\phi_j(x)}_{B(\phi_i, \phi_j)} + \phi_0$$

$$\boxed{w = \psi_i \neq \phi_i}$$

$$\int \underbrace{\psi_i}_{\leftarrow} (A \underbrace{\phi_j}_{\leftarrow}) dx = \int \psi_i f dx - \int \psi_i A(\phi_0) dx$$

$$\int \underline{\psi_i} [A(\phi_j, c_j)] dx = \int \psi_i f dx - \int \psi_i A(\phi_0) dx$$

Linear A  $\rightarrow$

$$\boxed{\underline{A_{ij}} c_j = F_i}$$

$\leftarrow$  Petrov-Galerkin's Method.

$$\underbrace{w}_{\leftarrow} = \psi_i = \phi_i$$

Linear A

$$\int \phi_i [A(\phi_j, c_j)] dx = \int \phi_i f dx - \int \phi_i A(\phi_0) dx$$

$$\boxed{\underline{A_{ij}} c_j = F_i}$$

$\leftarrow$  Galerkin's Method.

$$\boxed{\int_{\Omega} \phi_i R(\phi_j, c_j) d\Omega = 0}$$

$$w = \frac{\partial R}{\partial c_i} = \psi_i \rightarrow$$

$$\hat{u} = \sum_{j=1}^n c_j \phi_j(x) + \underline{\underline{\phi_0}}$$

Least Square Method

$$\underline{A\hat{u} - f = R}$$

$$\int R^2 d\Omega$$

$$\int \frac{\partial R}{\partial c_i} R d\Omega = 0$$

$$\Rightarrow \int \psi_i R d\Omega = 0$$

$$\sum_{j=1}^n \left[ \int A(\phi_i) A(\phi_j) dx \right] = \int_{\Omega} A(\phi_i) f dx - \int_{\Omega} A(\phi_i) \cdot \underline{A(\phi_0)} \quad i=1, \dots, n$$

$$\underline{A_{ij} c_j = F_j}$$

$$\underline{R_i = 0}$$

Collocation Method:

$$\psi_i = \delta(x - x^i)$$

$$\int_{\Omega} \delta(x - x^i) R_i dx = 0$$

$$\Rightarrow \int_{\Omega} \delta(x - x^i) R(x_j, c_j) dx = 0$$

$$Ax = b$$

$$\rightarrow A = \frac{1}{2} (Ax - b)^T (Ax - b)$$

$$\frac{\partial A}{\partial x} = A^T (Ax - b) = 0$$

$$\underline{A^T A x = A^T b}$$

$$\downarrow$$

$$Ax = b$$



$$\int_{\Omega} f(x) \delta(x - x^i) dx = f(x^i)$$

$$\left. \frac{du}{dz} \right|_{z=1} = 1$$



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**Module 03: Weighted Residual Method and 1D Bar Problem**

**Lecture 13 : Weighted Residual Method - Example**

# Concepts Covered

- Example Problem

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# Example:

Solve:  $\left(-\frac{d^2u}{dx^2} - u + x^2\right) = 0 \quad 0 < x < 1 \quad u(0) = 0$

Dirichlet's BC

Neumann's BC

$$\left.\frac{du}{dx}\right|_{x=1} = u'(1) = 1$$

Exact Solution:  $u_{Exact}(x) = \frac{2 \cos(1-x) - \sin(x)}{\cos(1)} + x^2 - 2$

n Parameter Approximation:  $u(x) = \sum_{j=1}^n c_j \phi_j + \phi_0$

2 Parameter Approximation:  $u(x) = c_1 \phi_1 + c_2 \phi_2 + \phi_0$

# Example:

2 Parameter Approximation:

$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0$$

$$u(0) = 0 \quad u'(1) = 1$$

Rayleigh Ritz Approximation:

Weak Form:  $B(w, u) = l(w) \Rightarrow \int_0^1 \left( \frac{dw}{dx} \frac{du}{dx} - wu \right) dx = - \int_1^1 wx^2 dx + w(1)$

$$\phi_0 = 0 \quad \phi_1 = x \quad \phi_2 = x^2 \quad u(x) = c_1x + c_2x^2$$

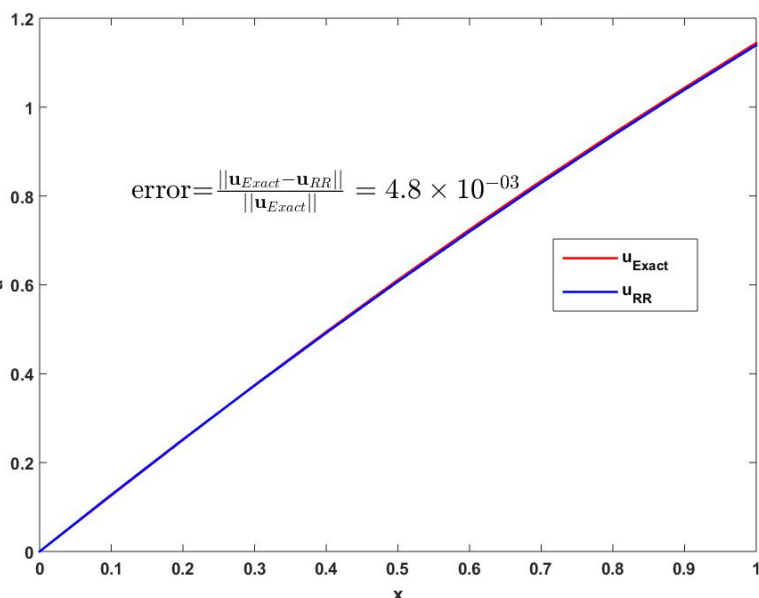
$$w(x) = \phi_i \quad w(x) = x \quad w(x) = x^2$$

$$\int_0^1 \left( \frac{d\phi_1}{dx} \left( c_1 \frac{d\phi_1}{dx} + c_2 \frac{d\phi_2}{dx} \right) - \phi_1(c_1x + c_2x^2) \right) dx = - \int_1^1 \phi_1 x^2 dx + \phi_1(1)$$

$$\int_0^1 \left( \frac{d\phi_2}{dx} \left( c_1 \frac{d\phi_1}{dx} + c_2 \frac{d\phi_2}{dx} \right) - \phi_2(c_1x + c_2x^2) \right) dx = - \int_1^1 \phi_2 x^2 dx + \phi_2(1)$$

# Example:

$$\begin{aligned} c_1 &= \mathbf{1.2905} \\ c_2 &= \mathbf{-0.1511} \end{aligned} \Leftrightarrow \begin{cases} \int_0^1 \left( \frac{d\phi_1}{dx} \left( c_1 \frac{d\phi_1}{dx} + c_2 \frac{d\phi_2}{dx} \right) - \phi_1(c_1 x + c_2 x^2) \right) dx = - \int_1^1 \phi_1 x^2 dx + \phi_1(1) \\ \int_0^1 \left( \frac{d\phi_2}{dx} \left( c_1 \frac{d\phi_1}{dx} + c_2 \frac{d\phi_2}{dx} \right) - \phi_2(c_1 x + c_2 x^2) \right) dx = - \int_1^1 \phi_2 x^2 dx + \phi_2(1) \end{cases}$$



$$u(x) = \mathbf{1.2905}x + \mathbf{-0.1511}x^2$$

$$B(w, u) = \mathbf{l(w)} \Rightarrow \int_0^1 \left( \frac{dw}{dx} \frac{du}{dx} - wu \right) dx = - \int_1^1 wx^2 dx + w(1)$$

$$\begin{aligned} & \frac{1}{2} [B(u, u) - \mathbf{l(u)}] \\ &= \frac{1}{2} \int_0^1 \left( \left( \frac{du}{dx} \right)^2 - u^2 + 2x^2 u \right) dx - u(1) \end{aligned}$$

# Example:

2 Parameter Approximation:

$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0$$

$$u(0) = 0 \quad u'(1) = 1$$

Galerkin's Approximation:  $w(x) = \phi_i$

$$\left. \begin{array}{l} \phi_0(0) = 0 \quad \phi'_0(1) = 1 \\ \phi_i(0) = 0 \quad \phi'_i(1) = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \phi_0(x) = x \\ \phi_1(x) = x(2-x) \quad \phi_2(x) = x^2 \left(1 - \frac{2}{3}x\right) \end{array} \right.$$

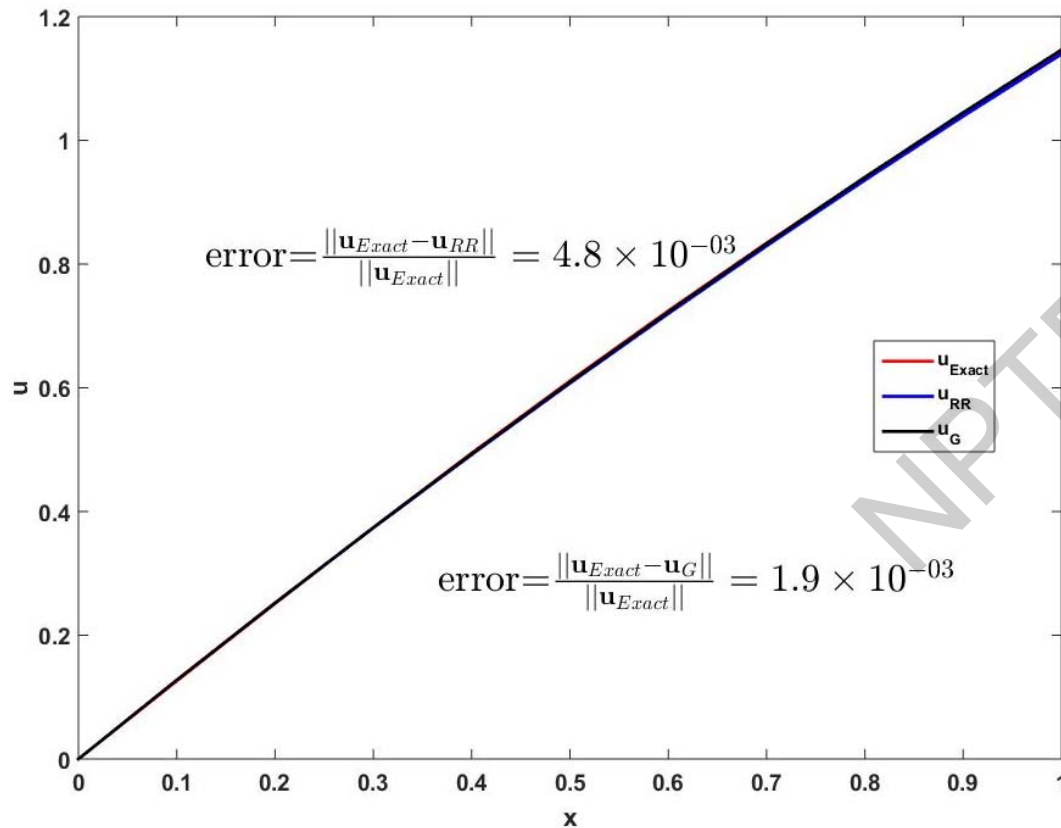
$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0 = c_1x(2-x) + c_2x^2 \left(1 - \frac{2}{3}x\right) + x$$

$$R = -\frac{d^2u}{dx^2} - u + x^2 = -\left(c_1\frac{d^2\phi_1}{dx^2} + c_2\frac{d^2\phi_2}{dx^2} + \frac{d^2\phi_0}{dx^2}\right) - (c_1\phi_1 + c_2\phi_2 + \phi_0) + x^2$$

$$\int_0^1 \phi_1 R \, dx = 0 \quad \int_0^1 \phi_2 R \, dx = 0 \quad \Rightarrow \quad \int_0^1 x(2-x)R \, dx = 0 \quad \int_0^1 x^2 \left(1 - \frac{2}{3}x\right) R \, dx = 0$$

$$u(x) = 0.1447x(2-x) + 0.0049x^2 \left(1 - \frac{2}{3}x\right) + x$$

# Example:



$$u_G(x) = 0.1447x(2 - x) + 0.0049x^2 \left(1 - \frac{2}{3}x\right) + x$$

$$u_{RR}(x) = 1.2905x + -0.1511x^2$$

# Example:

2 Parameter Approximation:

$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0$$

$$u(0) = 0 \quad u'(1) = 1$$

Petrov-Galerkin's Approximation:  $w(x) = \psi_i$

$$\left. \begin{array}{ll} \phi_0(0) = 0 & \phi'_0(1) = 1 \\ \phi_i(0) = 0 & \phi'_i(1) = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{ll} \phi_0(x) = x & \phi_2(x) = x^2 \left(1 - \frac{2}{3}x\right) \\ \phi_1(x) = x(2-x) & \end{array} \right.$$

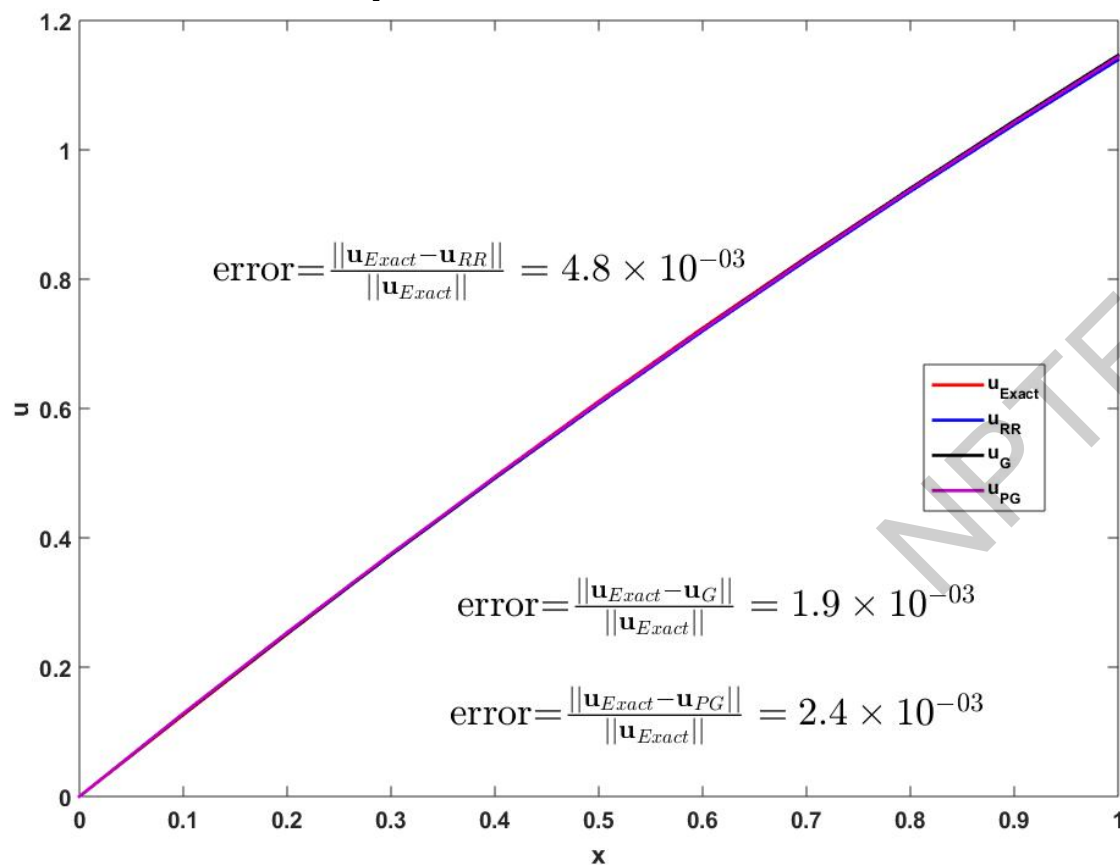
$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0 = c_1x(2-x) + c_2x^2\left(1 - \frac{2}{3}x\right) + x$$

$$\psi_1 = x \quad \psi_2 = x^2$$

$$\int_0^1 \psi_1 R \, dx = 0 \quad \int_0^1 \psi_2 R \, dx = 0 \quad \Rightarrow \quad \int_0^1 x R \, dx = 0 \quad \int_0^1 x^2 R \, dx = 0$$

$$u(x) = 0.1510x(2-x) - 0.022x^2\left(1 - \frac{2}{3}x\right) + x$$

# Example:



$$u_G(x) = 0.1447x(2 - x) + 0.0049x^2 \left(1 - \frac{2}{3}x\right) + x$$

$$u_{PG}(x) = 0.1510x(2 - x) - 0.022x^2 \left(1 - \frac{2}{3}x\right) + x$$

$$u_{RR}(x) = 1.2905x + -0.1511x^2$$

# Example:

2 Parameter Approximation:

$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0$$

$$u(0) = 0 \quad u'(1) = 1$$

Least Square Approximation:  $w(x) = \psi_i = \partial R / \partial c_i$

$$\left. \begin{array}{ll} \phi_0(0) = 0 & \phi'_0(1) = 1 \\ \phi_i(0) = 0 & \phi'_i(1) = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{ll} \phi_0(x) = x & \phi_2(x) = x^2 \left(1 - \frac{2}{3}x\right) \\ \phi_1(x) = x(2-x) & \end{array} \right.$$

$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0 = c_1x(2-x) + c_2x^2\left(1 - \frac{2}{3}x\right) + x$$

$$\psi_1 = \frac{\partial R}{\partial c_1} = 2 - 2x + x^2$$

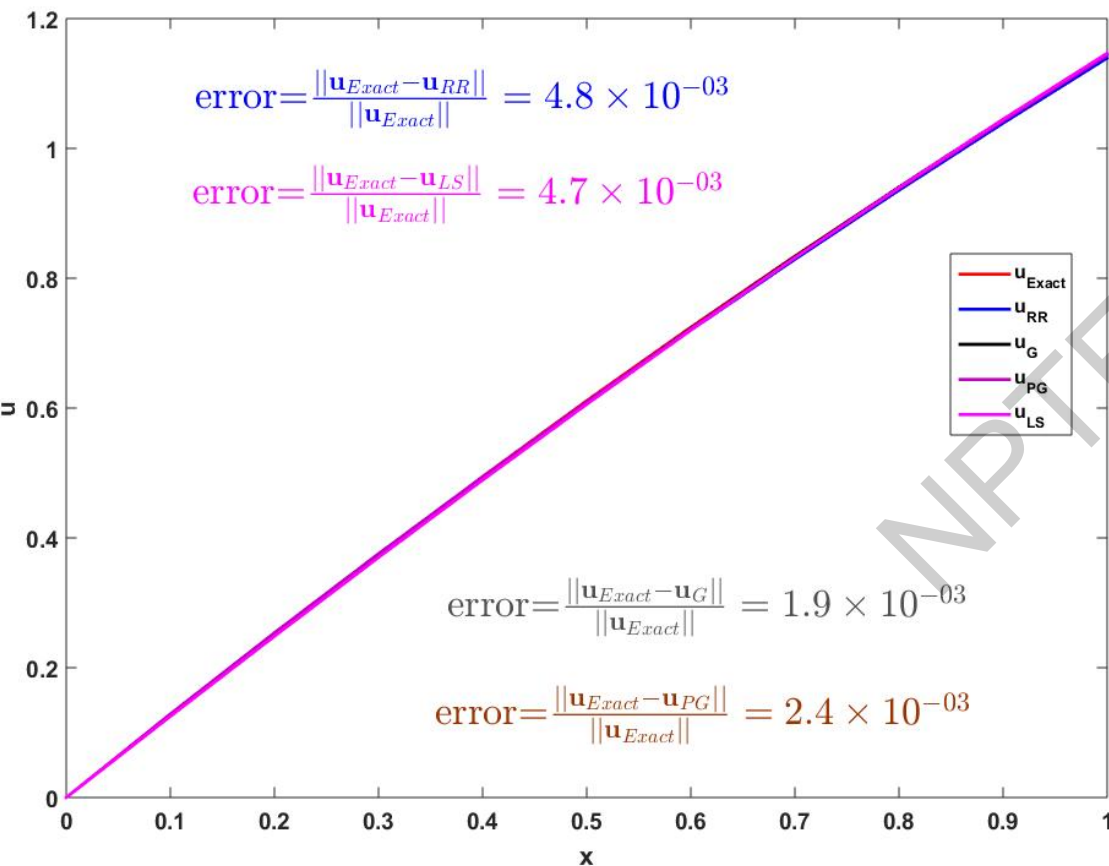
$$\psi_2 = \frac{\partial R}{\partial c_2} = -2 + 4x - x^2 + 2 - 2x + \frac{2}{3}x^3$$

$$\int_0^1 \psi_1 R \, dx = 0$$

$$\int_0^1 \psi_2 R \, dx = 0$$

$$u(x) = 0.13x(2-x) - 0.0499x^2\left(1 - \frac{2}{3}x\right) + x$$

# Example:



$$u_G(x) = 0.1447x(2-x) + 0.0049x^2 \left(1 - \frac{2}{3}x\right) + x$$

$$u_{PG}(x) = 0.1510x(2-x) - 0.022x^2 \left(1 - \frac{2}{3}x\right) + x$$

$$u_{LS}(x) = 0.13x(2-x) - 0.0499x^2 \left(1 - \frac{2}{3}x\right) + x$$

$$u_{RR}(x) = 1.2905x + -0.1511x^2$$



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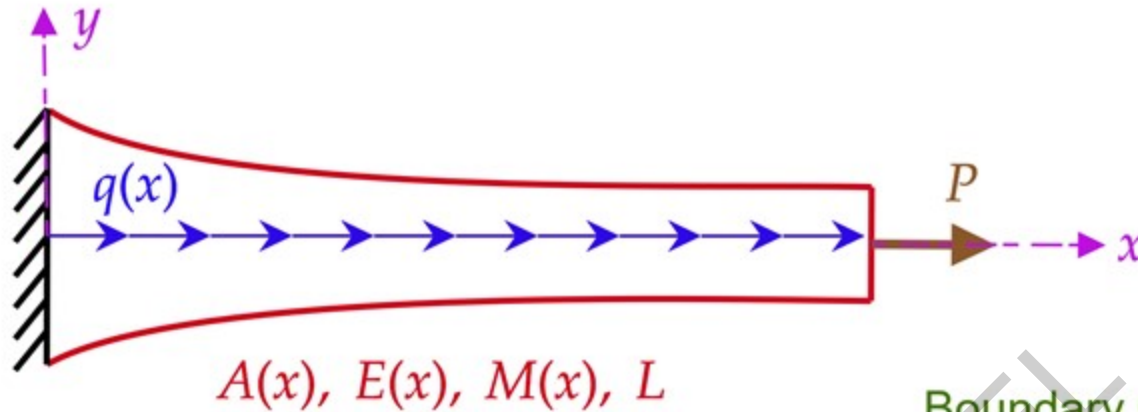
**Module 03: Weighted Residual Method and 1D Bar Problem**

**Lecture 14 : Concept of Element and Axial Bar Problem**

# Concepts Covered

- Axial Bar – Strong Form and Weak Form
- Eigen Value Problem
- Finite Elements Discretization
- Elementwise Weak Form
- Stiffness Matrix

# Axial Bar: Strong Form



Boundary Conditions

Boundary Value Problem: 
$$-\frac{d}{dx} \left( AE \frac{du}{dx} \right) = q \quad u(0) = 0 \quad \left( AE \frac{du(x)}{dx} \right)_{x=L} = P$$

Initial Value Problem: 
$$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{d}{dx} \left( AE \frac{\partial u}{\partial x} \right) = q \quad u(0, t) = 0 \quad \left( AE \frac{\partial u(x, t)}{\partial x} \right)_{x=L} = P$$

Initial Condition: 
$$u(x, 0) = 0 \quad \frac{\partial u(x, 0)}{\partial t} = \dot{u}(x, 0) = 0$$

## Axial Bar: Weak Form

*Weak Form*

$$\int_0^L \left( -\frac{d}{dx} \left( AE \frac{du}{dx} \right) - q \right) w \, dx = 0$$

$$\Rightarrow \int_0^L \left( \frac{dw}{dx} AE \frac{du}{dx} \right) - w(L) \left( AE \frac{du}{dx} \right)_{x=L} + \int_0^L q w \, dx = 0 \Rightarrow B(w, u) = l(w)$$

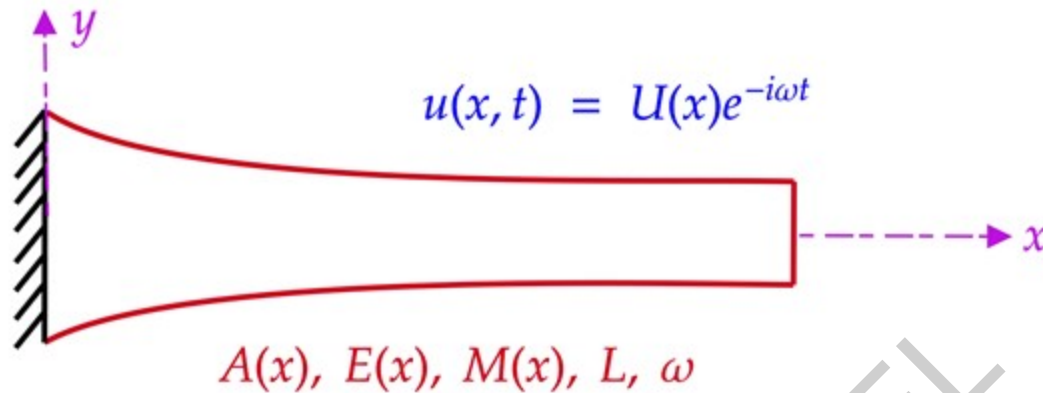
*Weak Form (dynamic case : Semi-discrete form)*

$$\int_0^L \left( \rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left( AE \frac{\partial u}{\partial x} \right) - q \right) w \, dx = 0$$

$$\Rightarrow \int_0^L \left( \frac{\partial w}{\partial x} AE \frac{\partial u}{\partial x} \right) dx + \int_0^L \left( w \rho A \frac{\partial^2 u}{\partial t^2} \right) dx - \int_0^L q w \, dx - w(L) \left( AE \frac{\partial u}{\partial x} \right)_{x=L} = 0$$



# Axial Bar: Eigen Value Problem

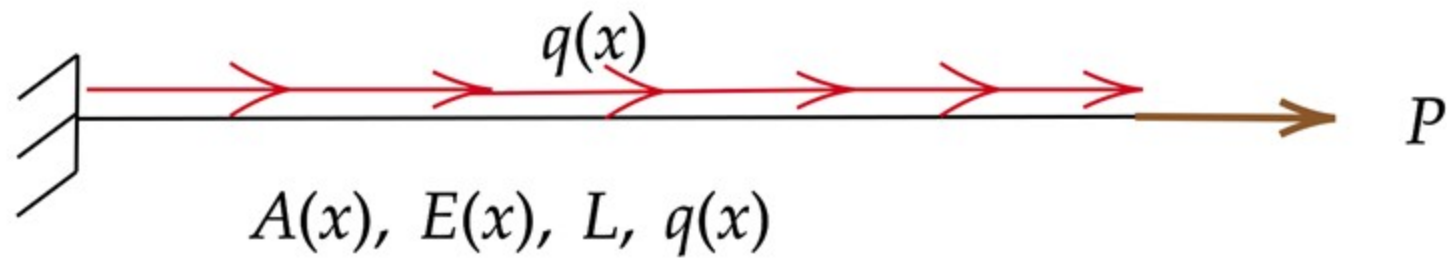
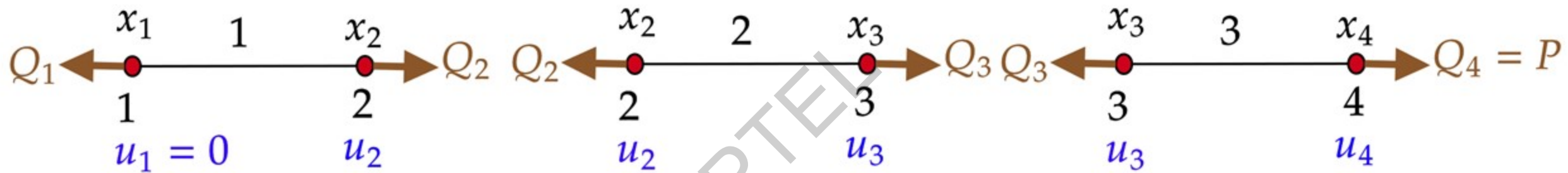
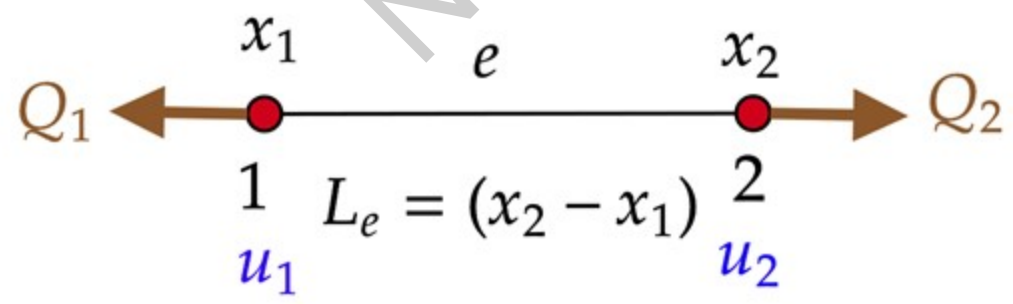


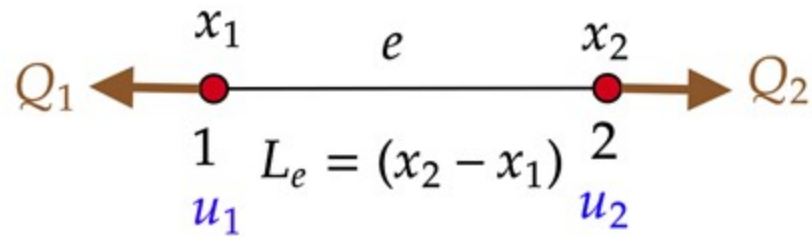
Eigen Value Problem:  $\frac{d}{dx} \left( AE \frac{dU}{dx} \right) + \omega^2 \rho A U = 0 \quad U(0) = 0 \quad \left( AE \frac{dU(x)}{dx} \right)_{x=L} = 0$

*Weak Form*

$$\int_0^L \left( \frac{d}{dx} \left( AE \frac{dU}{dx} \right) + \rho A \omega^2 U \right) w \, dx = 0$$

$$\Rightarrow - \int_0^L \left( \frac{dw}{dx} AE \frac{dU}{dx} \right) dx + \int_0^L w \rho A \omega^2 U dx + w(L) \left( AE \frac{dU}{dx} \right)_{x=L} = 0$$





$$u = \sum_{j=1}^2 \phi_j^e(x) u_j^e = \phi_1^e u_1^e + \phi_2^e u_2^e$$

Elemental Weak Form

$$\int_{x_1}^{x_2} \left[ -\frac{d}{dx} \left( AE \frac{d(\phi_1^e u_1^e + \phi_2^e u_2^e)}{dx} \right) - q \right] \phi_i dx = 0 \quad \text{for } i = 1, 2$$

$$\Rightarrow \int_{x_1}^{x_2} \left( \frac{d\phi_i}{dx} AE \frac{d\phi_1^e}{dx} \right) u_1^e dx + \int_{x_1}^{x_2} \left( \frac{d\phi_i}{dx} AE \frac{d\phi_2^e}{dx} \right) u_2^e dx - \int_{x_1}^{x_2} q \phi_i dx$$

$$- \phi_i(x_1) \left( AE \frac{du}{dx} \right)_{x=x_1} + \phi_i(x_2) \left( AE \frac{du}{dx} \right)_{x=x_2} = 0$$

$$\Rightarrow B(\phi_i, \phi_1) u_1 + B(\phi_i, \phi_2) u_2 = f(\phi_i) - \phi_i(x_1) Q(x_1) - \phi_i(x_2) Q(x_2) \quad \text{for } i = 1, 2$$

$$\Rightarrow \sum_{j=1}^2 B_{ij} u_j = f_i - \sum_{j=1}^2 \phi_i(x_j) Q_j \quad \text{for } i = 1, 2$$



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## **FINITE ELEMENT METHOD**

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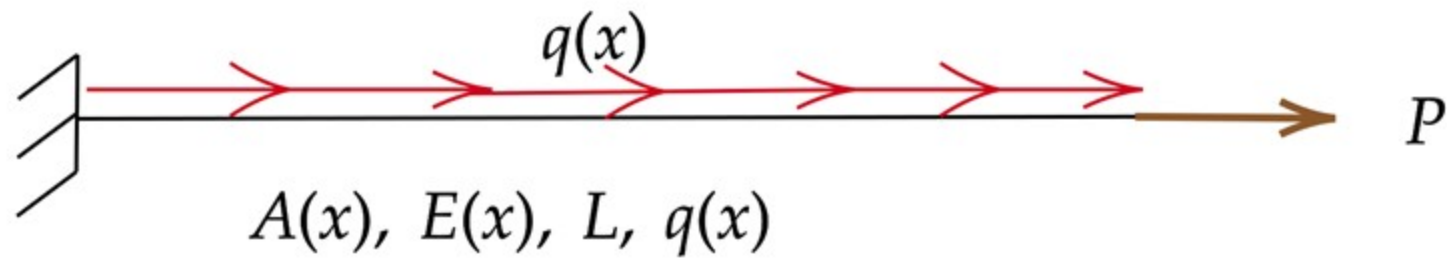
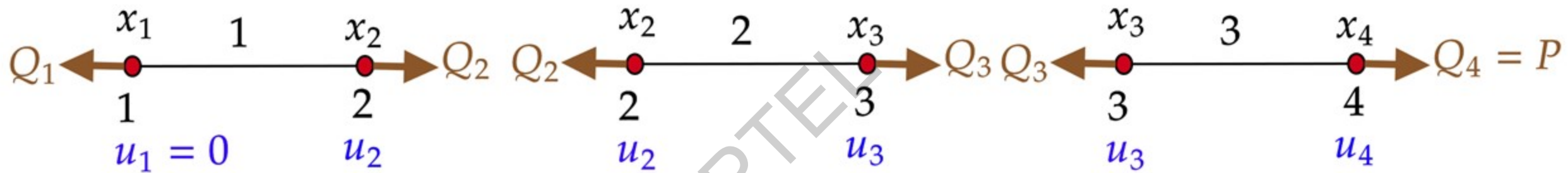
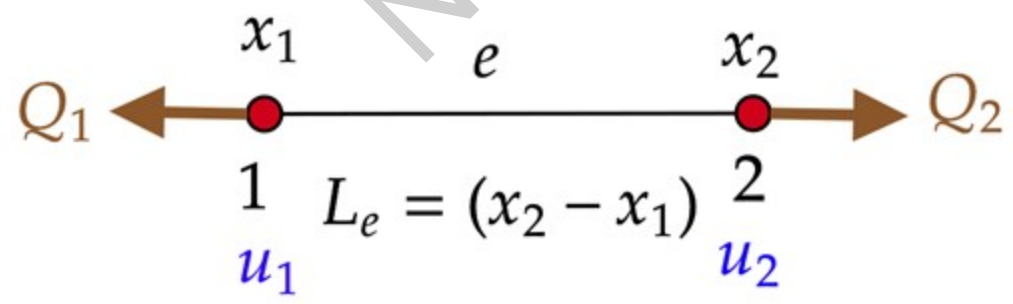
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**Module 03: Weighted Residual Method and 1D Bar Problem**

**Lecture 15 : Axial Bar Problem**

# Concepts Covered

- Finite Elements Discretization
- Element stiffness Matrix
- Global Stiffness Matrix
- Solution







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## **FINITE ELEMENT METHOD**

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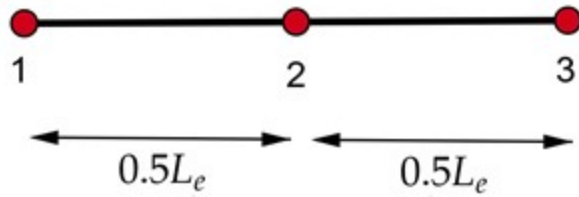
**Lecture 16 : Axial Bar - II**

# Concepts Covered

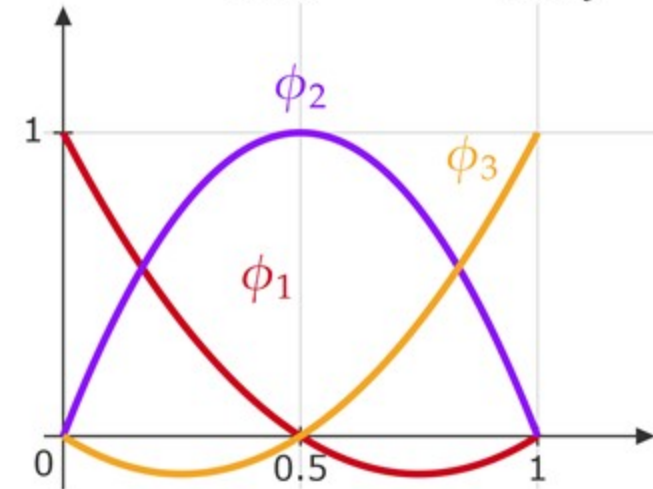
- Quadratic Shape Function
- Axial Bar Problem
- Eigen Value Problem

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## Quadratic Lagrange Shape Function



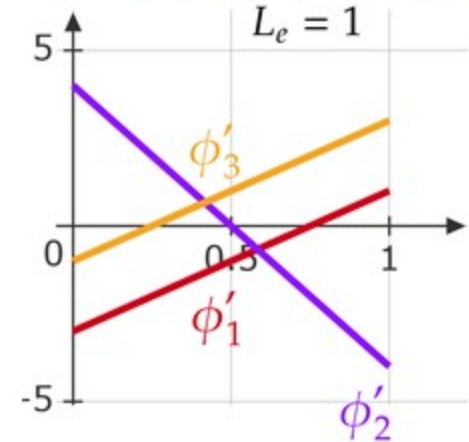
$$\phi_1 = \left(1 - \frac{\bar{x}}{L_e}\right) \left(1 - \frac{2\bar{x}}{L_e}\right)$$

$$\phi_2 = 4 \frac{\bar{x}}{L_e} \left(1 - \frac{\bar{x}}{L_e}\right)$$

$$\phi_3 = -\frac{\bar{x}}{L_e} \left(1 - \frac{2\bar{x}}{L_e}\right)$$

$$u = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$$

within element  $C^2$  continuous  
interelement boundary  $C^1$   
continuous



$$\phi'_1 = \left(\frac{4\bar{x}}{L_e^2} - \frac{3}{L_e}\right)$$

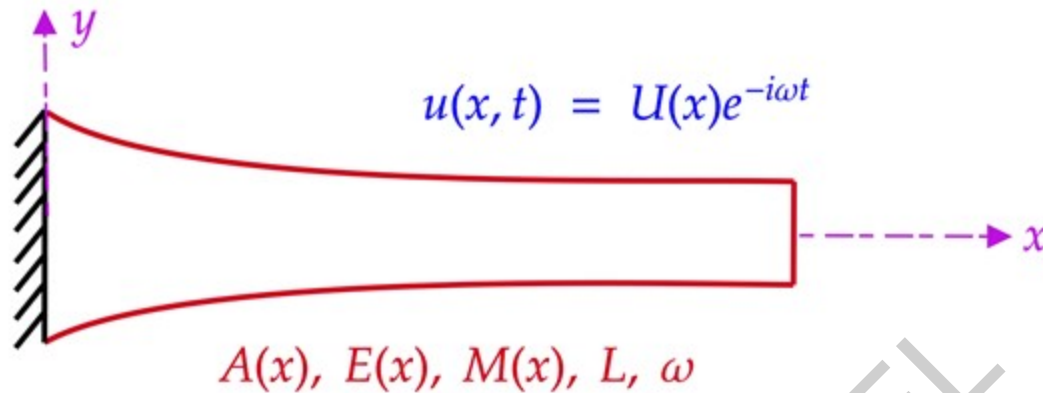
$$\phi'_2 = \frac{4}{L_e} - \frac{8\bar{x}}{L_e^2}$$

$$\phi'_3 = \left(\frac{4\bar{x}}{L_e^2} - \frac{1}{L_e}\right)$$

$$\frac{du}{d\bar{x}} = u_1\phi'_1 + u_2\phi'_2 + u_3\phi'_3$$

within element  $C^1$  continuous  
interelement boundary  $C^0$   
continuous

## Axial Bar: Eigen Value Problem

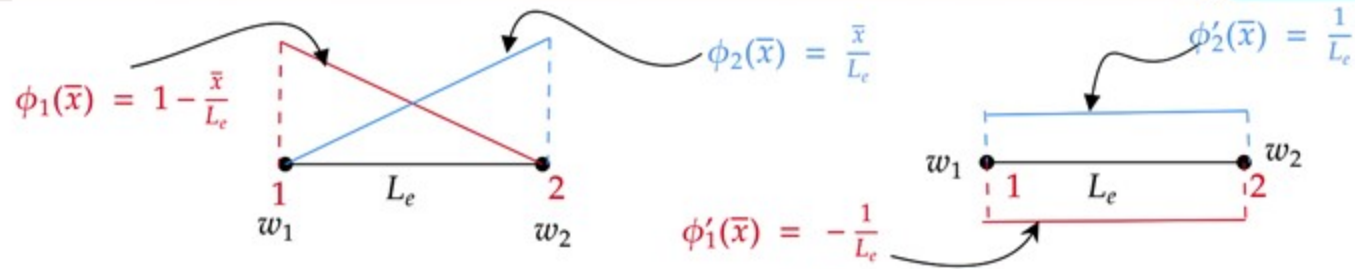


Eigen Value Problem:  $\frac{d}{dx} \left( AE \frac{dU}{dx} \right) + \omega^2 \rho A U = 0 \quad U(0) = 0 \quad \left( AE \frac{dU(x)}{dx} \right)_{x=L} = 0$

*Weak Form*

$$\int_0^L \left( \frac{d}{dx} \left( AE \frac{dU}{dx} \right) + \rho A \omega^2 U \right) w \, dx = 0$$
$$\Rightarrow - \int_0^L \left( \frac{dw}{dx} AE \frac{dU}{dx} \right) dx + \int_0^L w \rho A \omega^2 U dx + w(L) \left( AE \frac{dU}{dx} \right)_{x=L} = 0$$

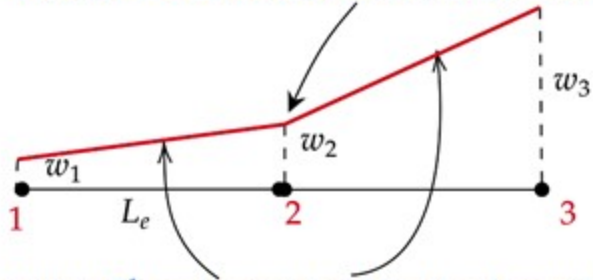
# Linear Shape Function



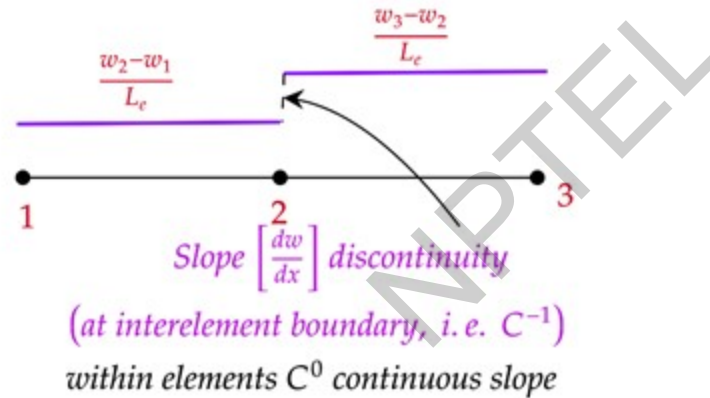
$$w(\bar{x}) = \phi_1(\bar{x})w_1 + \phi_2(\bar{x})w_2$$

$$\begin{aligned} \frac{dw(\bar{x})}{dx} &= \frac{d\phi_1(\bar{x})}{dx}w_1 + \frac{d\phi_2(\bar{x})}{dx}w_2 = \phi_1'(\bar{x})w_1 + \phi_2'(\bar{x})w_2 \\ &= \frac{w_2 - w_1}{L_e} \text{ (Constant)} \end{aligned}$$

$w(x)$  is  $C^0$  continuous (at interelement boundary)



$w(x)$  is  $C^1$  continuous (within the element)



$w(x)$  is  $C^1$  continuous function within the element but  $C^0$  continuous at interelement boundary  
 $\frac{dw(x)}{dx}$  is  $C^0$  continuous function within the element but discontinuous or  $C^{-1}$  at the interelement boundary

So, this element is  $C^0$  element because it can give at least  $C^0$  global continuity of the field variable



**THANK  
YOU !**